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Another proof of Soittola's theorem

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Abstract

Soittola's theorem characterizes $\mathbb{R}_+$- or $\mathbb{N}$-rational formal power series in one variable among the rational formal power series with nonnegative coefficients. We present here a new proof of the theorem based on Soittola's and Perrin's proofs together with some new ideas that allows us to separate algebraic and analytic arguments.

Keywords: Formal power series; Rational series; Poles of rational fractions

1. Introduction

Soittola's theorem characterizes $\mathbb{R}_+$- or $\mathbb{N}$-rational formal power series in one variable among the rational formal power series with nonnegative coefficients. Recently, there was a renewal of interest in these series, both for the combinatorial aspect [7] and for the computational problems, such as reverse engineering [2] and effective implementation of the algorithm underlying Soittola's proof [11,12]. The importance of these series comes, among others, from the fact that $\mathbb{N}$-rational series are precisely the generating series of rational languages. It is a remarkable property of these series that they admit also an analytic characterization by their poles.

The aim of this paper is to present a new proof of Soittola's theorem, that is a merge of Soittola's original proof, of Perrin's proof and of some new ideas that will allow us to separate algebraic and analytic arguments.

The authors developed this proof when they went through the manuscript of their book [4] during the process of preparing a new edition.

Recall that an $\mathbb{N}$-rational series in the variable x is obtained by applying sums, products and the star operation $S^* = \sum_{n\geq 0} S^n$ (where S has 0 constant term), starting with polynomials over $\mathbb{N}$. For example, the Fibonacci series $\sum_{n\geq 0} F_n x^n$, with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ is $\mathbb{N}$-rational since it is equal to $(x + x^2)^*$. Also, the series $3x^2 + ((x + 3x^2)^* x + x^4)^*$ is another example of an $\mathbb{N}$-rational series.

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Our main result is a new proof of Soittola's characterization (Theorem 2.1) of K_+-rational series in one variable when $K = \mathbb{Z}$ or K is a subfield of $\mathbb{R}$.

The star height of positive series is the concern of the last section. We prove the result of [10] (Theorem 3.1): each K_+-rational series in one variable has star height at most 2, and has a rational expression of a very special form; in particular, the argument of the innermost star is a monomial. This result is implicit also in Soittola's paper.

Consider series of the form

$$\sum a_n x^n$$

with all coefficients in $\mathbb{R}_+$. If such a series is the expansion of a rational function, it does not imply in general that it is $\mathbb{R}_+$-rational (examples are given in [3], [5] Example VIII.6.1, [16] Exercise II.10.2, [4] Exercise V.2.2). We shall characterize in the next section those rational functions over $\mathbb{R}$ whose series expansion is $\mathbb{R}_+$-rational. We call them *$\mathbb{R}_+$-rational functions*. The necessary condition is given by:

Theorem 1.1 ([3]). *Let $f(x)$ be an $\mathbb{R}_+$-rational function which is not a polynomial, and let ρ be the minimum of the moduli of its poles. Then ρ is a pole of f, and any pole of f of modulus ρ has the form $\rho\theta$, where θ is a root of unity.*

Observe that the minimum of the moduli of the poles of a rational function is just the radius of convergence of the associated series.

We recall, for later use, the following weak converse.

Proposition 1.2. *Let $S = \sum a_n x^n$ be a $\mathbb{Z}$-rational series which has polynomial growth. If the coefficients a_n are in $\mathbb{N}$, then S is $\mathbb{N}$-rational.*

This proposition is Exercise II.10.3 in [16] or Proposition VIII.2 in the electronic version of [4].

2. Characterization

Theorem 1.1 gives a necessary condition for a rational function to be $\mathbb{R}_+$-rational. We now give a sufficient condition in the general case.

Let $S = \sum_{n\geq 0} a_n x^n$ be a rational series which is not a polynomial. It is well-known that there exists an *exponential polynomial* for a_n, that is

$$a_n = \sum_i P_i(n)\lambda_i^n \tag{1}$$

for n large enough, where the P_i's are polynomials over $\mathbb{C}$ and the λ_i are nonzero complex numbers. The λ_i are called the *eigenvalues* of S (in [4] they were called the *roots* of S). The *multiplicity* of λ_i is $\deg(P_i) + 1$. The eigenvalues are the inverses of the poles of the rational fraction associated with S, with the same multiplicity.

A rational series with complex coefficients which is not a polynomial is said to have a *dominating eigenvalue* if there is, among its eigenvalues, a unique eigenvalue having maximal modulus. It is equivalent to say that the associated rational function has a unique pole of minimal modulus.

Theorem 2.1 ([15]). *Let $K = \mathbb{Z}$ or K be a subfield of $\mathbb{R}$. A K-rational series with nonnegative coefficients which is not a polynomial and which has a dominating eigenvalue is K_+-rational.*

For the sake of completeness, we mention without proof (which is easy) the following complete characterization of K_+-rational series. Recall that the *merge* of series $S_0, \ldots, S_{p-1}$ is the series

$$\sum_{i=0}^{p-1} x^i S_i(x^p).$$

Corollary 2.2. *A series over K_+ is K_+-rational if and only if it is the merge of polynomials and of rational series having a dominating eigenvalue.*

Let $S = \sum_{n\geq 0} a_n x^n$ be a series which is not a polynomial and suppose, using (1), that λ_1 is the dominating eigenvalue of S. We call *dominating coefficient* of S the dominating coefficient α of P_1. Observe that when $n \to \infty$

$$a_n \sim \alpha n^{\deg(P_1)} \lambda_1^n \tag{2}$$

and

$$\frac{a_{n+1}}{a_n} \sim \lambda_1. \tag{3}$$

Lemma 2.3. *Let S, S' be real series which are not polynomials and which have the same dominating eigenvalue λ_1 with dominating coefficients α, α'.*

(i) *The series SS' has also the dominating eigenvalue λ_1 with dominating coefficient positively proportional to $\alpha\alpha'$.*

(ii) *The coefficients of S are ultimately positive if and only if λ_1 and α are positive real numbers.*

(iii) *If S is the inverse of a polynomial P with $P(0) = 1$, and if λ_1 is a positive real number, then α also is a positive real number.*

Proof. (i) We write S as a $\mathbb{C}$-linear combination of partial fractions. Let β be the coefficient of $1/(1-\lambda_1 x)^{k+1}$ in this combination, where $k = \deg(P_1)$. Since $1/(1-\lambda_1 x)^{k+1} = \sum_{n\geq 0} \binom{n+k}{k} \lambda_1^n x^n$ and $\binom{n+k}{k} = \frac{n^k}{k!} + \cdots$, the dominating term of $P_1(n)$ is $\beta \frac{n^k}{k!}$, and $\alpha = \beta/k!$. If we do similarly for S', we obtain a dominating term of the form $\beta' \frac{n^\ell}{\ell!}$ and $\alpha' = \beta'/\ell!$. The product SS' has the eigenvalue λ_1 with multiplicity $k+\ell+2$, the dominating term is $\beta\beta' \frac{n^{k+\ell+1}}{(k+\ell+1)!}$, so the dominating coefficient is $\alpha\alpha' k!\ell!/(k+\ell+1)!$. This gives the result.

(ii) If the a_n are ultimately positive, then $\lambda_1 > 0$ by (3). Moreover, α is positive by (2). Conversely, if $\lambda_1, \alpha > 0$, then $a_n > 0$ for n large enough by (2).

(iii) We have $P(x) = \prod_{i=1}^{d} (1-\lambda_i x) \in \mathbb{R}[x]$ with $\lambda_i \in \mathbb{C}$, $\lambda_1 = \cdots = \lambda_k > |\lambda_{k+1}|, \ldots, |\lambda_d|$, for some k with $1 \leq k \leq d$. In order to compute the dominating coefficient α of P^{-1}, we write P^{-1} as a $\mathbb{C}$-linear combination of series $1/(1-\lambda_i x)^j$. Then $\alpha = \beta/(k-1)!$ where β is the coefficient of $1/(1-\lambda_1 x)^k$ in this linear combination. To compute β, multiply the linear combination by $(1-\lambda_1 x)^k$ and put then $x = \lambda_1^{-1}$. Since only fractions $1/(1-\lambda_1 x)^j$ with $j \leq k$ occur, this is well defined and gives

$$\beta = \frac{1}{\prod_{i=k+1}^{d} \left(1 - \frac{\lambda_i}{\lambda_1}\right)}.$$

Now, the numbers λ_i^{-1}, for $i = k+1, \ldots, d$ are the roots of the real polynomial $\prod_{i=k+1}^{d} (1-\lambda_i x)$. Hence, either λ_i is real and then $|\lambda_i| < \lambda_1$ and thus $1 - \frac{\lambda_i}{\lambda_1} > 0$, or λ_i is not real and then there is some j such that λ_i, λ_j are conjugate. Then so are $1 - \frac{\lambda_i}{\lambda_1}$ and $1 - \frac{\lambda_j}{\lambda_1}$, so that their product is positive. Hence α is positive. $\square$

Given an integer $d \geq 1$ and numbers $B, G_1, \ldots, G_d$ in $\mathbb{R}_+$, we set

$$\begin{aligned} G(x) &= \sum_{i=1}^{d-1} G_i x^i, \\ D(x) &= (1-Bx)(1-G(x)) - G_d x^d. \end{aligned} \tag{4}$$

If $d = 1$, we agree that $B = 0$. In this limit case, $D(x) = 1 - G_1 x$. In view of [15], we call a polynomial $D(x)$ of the form (4) a *Soittola denominator*. The numbers $B, G_1, \ldots, G_d$ are called the *Soittola coefficients* of $D(x)$ and B is called its *modulus*.

Note that setting

$$D(x) = 1 - g_1 x - \cdots - g_d x^d$$

the Eq. (4) is equivalent to

$$\begin{aligned} g_1 &= B + G_1 \\ g_i &= G_i - BG_{i-1}, \quad i = 2, \ldots, d. \end{aligned} \tag{5}$$

Likewise, we call *Soittola polynomial* a polynomial of the form

$$x^d - g_1 x^{d-1} - \cdots - g_d \tag{6}$$

with the g_i as above. Thus a Soittola polynomial is the reciprocal polynomial of a Soittola denominator.

Lemma 2.4. *Let*

$$P(x) = \prod_{i=1}^{d} (1 - \lambda_i x)$$

be a polynomial in $\mathbb{R}[x]$ *with* $\lambda_i \in \mathbb{C}$, $\lambda_1 > 1$, *and* $\lambda_1 > |\lambda_2|, \ldots, |\lambda_d|$. *Let*

$$P_n(x) = \prod_{i=1}^{d} (1 - \lambda_i^n x).$$

For n large enough, $P_n(x)$ *is a Soittola denominator with modulus* $< \lambda_1^n$ *and with Soittola coefficients in the subring generated by the coefficients of* P.

Proof. The fundamental theorem of symmetric functions states that every symmetric polynomial in variables $\lambda_1, \ldots, \lambda_d$ over $\mathbb{Z}$ is a polynomial over $\mathbb{Z}$ in the elementary symmetric functions. See [13], pages 20–21.

Let $e_{i,n}$ be the ith elementary symmetric function of $\lambda_1^n, \ldots, \lambda_d^n$. We conclude that $e_{i,n}$ is in the ring generated by the $e_{i,1}$, hence in the ring generated by the coefficients of $P = P_1$.

Clearly $e_{1,n} \sim \lambda_1^n$ when $n \to \infty$. Note that for $i \geq 2$, each term in $e_{i,1}$ is a product of i factors taken in the λ_j's, and containing at least one factor with modulus $< \lambda_1$. Similarly for $e_{i,n}$. Therefore $e_{i,n}/\lambda_1^{in} \to 0$ when $n \to \infty$.

We may assume that $d \geq 2$. Define $B = \lfloor e_{1,n}/2 \rfloor$ and $G_1, \ldots, G_d$ by the formulas $G_1 = e_{1,n} - B$ and $G_i = BG_{i-1} + (-1)^{i-1} e_{i,n}$ for $i = 2, \ldots, d$ (we do not indicate the dependence on n which is understood). Since $\lambda_1^n \to \infty$, we have $B \sim \lambda_1^n/2 \sim G_1$. Arguing by induction on i, suppose that $G_i \sim \lambda_1^{in}/2^i$. We have $G_{i+1} = (-1)^i e_{i+1,n} + BG_i$. Now $BG_i \sim \lambda_1^{(i+1)n}/2^{i+1}$ and we know that $e_{i+1,n}/\lambda_1^{(i+1)n} \to 0$. Thus $G_{i+1} \sim \lambda_1^{(i+1)n}/2^{i+1}$. The lemma follows by (5), since when P_n is given the form (6), then we have $g_i = (-1)^{i-1} e_{i,n}$. $\square$

In view of [14], we call *Perrin companion matrix* of the Soittola polynomial (6) the matrix

$$P = \begin{pmatrix} B & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & & & \\ & & \ddots & & 1 & 0 \\ 0 & \cdots & & & 0 & 1 \\ G_d & & & & G_2 & G_1 \end{pmatrix}. \tag{7}$$

It differs from a usual companion matrix by the entry 1, 1 which is not 0 but B. In the limit case $d = 1$, one sets $P = (G_1)$.

Lemma 2.5. *Let* $D(x)$ *be the Soittola denominator* (4). *Given* $S = \sum a_n x^n$, *define* $T = \sum t_n x^n$ *and* $U = \sum u_n x^n$ *by*

$$T = DS \quad \text{and} \quad U = (1 - Bx)S.$$

Then for $n \geq 0$,

$$P \begin{pmatrix} a_n \\ u_{n+1} \\ \vdots \\ u_{n+d-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t_{n+d} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d} \end{pmatrix}. \tag{8}$$

Moreover, if T *is a polynomial of degree* $< h$, *then for any* n

$$a_{n+h} = (1, 0, \ldots, 0) P^n (a_h, u_{h+1}, \ldots, u_{h+d-1})^t.$$

Proof. Note that in the limit case $d = 1$, Eq. (8) must be read as $G_1 a_n + t_{n+1} = a_{n+1}$, which is easy to verify. We thus may assume that $d \geq 2$. The first matrix product is equal to

$$\begin{pmatrix} Ba_n + u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+d-1} \\ \alpha \end{pmatrix}$$

where

$$\alpha = G_d a_n + \sum_{i=1}^{d-1} G_i u_{n+d-i}.$$

Observe next that by (4)

$$T = (1 - Bx)(1 - G(x))S - G_d x^d S = (1 - G(x))U - G_d x^d S.$$

Thus

$$t_{n+d} = u_{n+d} - \sum_{i=1}^{d-1} G_i u_{n+d-i} - G_d a_n,$$

showing that $\alpha + t_{n+d} = u_{n+d}$. This proves the first identity. Suppose now that T is a polynomial of degree $< h$. Then $0 = t_{h+d} = t_{h+d+1} = \cdots$. Using induction and (8) for $n = h, h+1, \ldots$, we obtain

$$P^n \begin{pmatrix} a_h \\ u_{h+1} \\ \vdots \\ u_{h+d-1} \end{pmatrix} = \begin{pmatrix} a_{n+h} \\ u_{n+h+1} \\ \vdots \\ u_{n+h+d-1} \end{pmatrix}$$

which implies the second identity. □

Proof of Soittola's theorem. 1. By Lemma 2.3(ii), the dominating eigenvalue λ_1 of S is positive. We may assume that $\lambda_1 > 1$. Indeed, if K is a subfield of $\mathbb{R}$, then we replace $S(x)$ by $S(\alpha x)$ for α in $\mathbb{N}$ large enough; then the eigenvalues are multiplied by α and we are done. If $K = \mathbb{Z}$ and $\lambda_1 \leq 1$, then by Proposition 1.2, S is $\mathbb{N}$-rational.

2. Write $S(x) = N(x)/D(x)$ where D is the smallest denominator with $D(0) = 1$. Then $N, D \in K[x]$ (this follows from Fatou's lemma, see [4]). Let m be the multiplicity of the eigenvalue λ_1 of S. Since K is a factorial subring of $\mathbb{R}$, we may write $D(x) = D_1(x) \cdots D_m(x)$, where each polynomial $D_i(x)$ has coefficients in K, has the simple factor $1 - \lambda_1 x$ and satisfies $D_i(0) = 1$.

Decompose S as a merge $S = \sum_{0 \leq i < p} x^i S_i(x^p)$. Then the eigenvalues of S_i are the pth powers of those of S (equivalently the poles of S_i are the pth powers of those of S). Hence, if p is chosen large enough, Lemma 2.4 shows that we may assume that D_1 is a Soittola denominator of the form

$$D_1(x) = (1 - Bx)\left(1 - \sum_{i=1}^{d-1} G_i x^i\right) - G_d x^d$$

with $d \geq 1$, $B, G_i \in K_+$ and $B < \lambda_1$. Since $a_{n+1}/a_n \sim \lambda_1$ we see that $u_{n+1} = a_{n+1} - Ba_n \geq 0$ for n large enough.

3. Let

$$T = \sum_{n \geq 0} t_n x^n = D_1 S.$$

Suppose first that λ_1 is simple, that is $m = 1$. Then T is a polynomial and Lemma 2.5 shows that $\sum_{n \geq 0} a_{n+h} x^n$ is K_+-rational for h large enough. Hence S is K_+-rational. Suppose next that $m \geq 2$ and argue by induction on m. Note that S, D_1^{-1} and T have the dominating eigenvalue λ_1, the latter with multiplicity $m - 1$. Lemma 2.3(iii) and (ii) show that D_1^{-1} and S have positive dominating coefficient. Thus by Lemma 2.3(i), since $D_1^{-1} T = S$, the series T also has positive dominating coefficient. This implies that T has ultimately positive coefficients and thus that for h large enough, the series $\sum_{n\geq 0} t_{n+h+d}x^n$ is K_+-rational, by induction on m.

Thus $t_{n+h+d} = \nu N^n \gamma$ for some representation (ν, N, γ) over K_+. Define a representation (ℓ, M, c) over K_+ by

$$\ell = (1, 0, \ldots, 0), \quad M = \begin{pmatrix} P & Q \\ 0 & N \end{pmatrix}, \quad c = \begin{pmatrix} a_h \\ u_{h+1} \\ \vdots \\ u_{h+d-1} \\ \gamma \end{pmatrix}$$

where h is chosen large enough and where all rows of Q are 0 except the last which is ν. We prove that

$$M^n c = \begin{pmatrix} a_{h+n} \\ u_{h+n+1} \\ \vdots \\ u_{h+n+d-1} \\ N^n\gamma \end{pmatrix}.$$

This is true for $n = 0$ by definition. Admitting it holds for n, the equality for $n + 1$ follows from Lemma 2.5 (where n is replaced by $n + h$), since $QN^n\gamma$ is a column vector whose components are all 0 except the last one which is $\nu N^n \gamma = t_{n+h+d}$. We deduce that $\ell M^n c = a_{n+h}$ and $S = \sum_{i=0}^{h-1} a_i x^i + x^h \sum_{n\geq 0} a_{n+h}x^n$ is therefore K_+-rational. $\square$

3. Series of star height 2

We consider now the star height of K_+-rational series.

Theorem 3.1. *Let K be a subfield of $\mathbb{R}$ or $K = \mathbb{Z}$. Any K_+-rational series is in the subsemiring of $K_+[[x]]$ generated by $K_+[x]$ and by the series of the form*

$$(Bx^p)^* \quad or \quad \left(\sum_{i=1}^{d-1} G_i x^i + G_d x^d (Bx^p)^*\right)^*$$

with $p, d \geq 1$, $B, G_i \in K_+$. In particular, they have star height at most 2.

Note that there exist $\mathbb{R}_+$-rational series of star height 2. For instance, the series $(2x + x^2x^*)^* = \sum_{n\geq 0} F_{2n}x^n$ has star height 2 [1]. A related result is in [9].

As another example, consider the sequence $f^{(n)}$ of $\mathbb{N}$-rational series given by $f^{(0)} = 1$, and $f^{(n+1)} = (xf^n)^*$. For example,

$$f^{(5)} = (x(x(x(xx^*)^*)^*)^*)^* = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1-x}}}}}.$$

Then $f^{(5)} = \frac{1-4x+3x^2}{1-5x+6x^2-x^3}$ and the argument of the proof below shows that

$$f^{(5)} = 1 + x(1 + x^2(3x)^*)(2x + x^3(3x)^*)^*.$$

Proof of Theorem 3.1. Denote by $\mathcal{L}$ the semiring defined in the statement. It is clearly closed under the substitution $x \mapsto \alpha x^q$ for $q \geq 1, \alpha \in K_+$. Thus it is also closed under the merge of series.

So, if we follow the proof of Soittola's theorem, we may pursue after steps (1) and (2). We start with a notation. Given a series $V = \sum_{n\geq 0} v_n x^n$ and an integer $h \geq 0$, we write $V^{(h)} = \sum_{n>h} v_n x^n$ and $V_{(h)} = \sum_{n\leq h} v_n x^n$. Thus it follows from $U = (1 - Bx)S$ that

$$U^{(h)} = S^{(h)} - BxS^{(h-1)} = S^{(h)}(1 - Bx) - Ba_h x^{h+1}$$
$$U_{(h)} = S_{(h)} - BxS_{(h-1)} = S_{(h-1)}(1 - Bx) + a_h x^h.$$

We show below the existence of a polynomial P_h with coefficients in K_+, for h large enough, such that

$$U^{(h)} = \left(P_h + T^{(h)} + a_h G_d x^{h+d}(Bx)^*\right)H^*$$

where

$$H = G + G_d x^d (Bx)^*.$$

If $m = 1$, we take h large enough and $T^{(h)} = 0$. If $m \geq 2$, we conclude by induction on m that $T^{(h)}$ is in $\mathcal{L}$. Thus the series $U^{(h)}$ is in $\mathcal{L}$, and since $(1 - Bx)S^{(h)} = Ba_h x^{h+1} + U^{(h)}$ the series

$$S = \sum_{i=0}^{h} a_i x^i + (Bx)^*(Ba_h x^{h+1} + U^{(h)})$$

is in $\mathcal{L}$.

Now from

$$T = D_1 S = (1 - Bx)(1 - H)S = U(1 - H),$$

we get

$$\begin{aligned} T^{(h)} &= \big(U(1-H)\big)^{(h)} = \big(U^{(h)}(1-H)\big)^{(h)} + \big(U_{(h)}(1-H)\big)^{(h)} \\ &= U^{(h)}(1-H) + \big(U_{(h)} - U_{(h)}H\big)^{(h)} \\ &= U^{(h)}(1-H) - \big(U_{(h)}H\big)^{(h)}. \end{aligned}$$

Next

$$\big(U_{(h)}H\big)^{(h)} = \big(U_{(h)}G\big)^{(h)} + \big(U_{(h)}G_d x^d (Bx)^*\big)^{(h)}.$$

Recall that $G = \sum_{i=1}^{d-1} G_i x^i$. The first term on the right-hand side is

$$\big(U_{(h)}H\big)^{(h)} = \sum_{\substack{0\leq j\leq h\\ 0<\ell<d\\ j+\ell>h}} u_j G_\ell x^{j+\ell}.$$

Setting $j + \ell = h + i$ with $0 < i < d$, this rewrites as $\sum_{i=1}^{d-1} w_i x^{h+i}$ with

$$w_i = \sum_{\substack{0\leq j\leq h\\ 0<\ell<d\\ j+\ell=h+i}} u_j G_\ell.$$

Now note that in this sum, since $\ell < d$, we have $j > h - d$, hence $u_j \geq 0$ for h large enough. This shows that $\big(U_{(h)}H\big)^{(h)}$ is a polynomial with coefficients in K_+.

To compute the second term, recall that $U_{(h)} = S_{(h-1)}(1 - Bx) + a_h x^h$. Consequently

$$U_{(h)}(Bx)^* = S_{(h-1)} + a_h x^h (Bx)^*.$$

So the term $(U_{(h)}G_d x^d (Bx)^*)^{(h)}$ reduces to the sum of a polynomial with coefficients in K_+ and of the series $G_d a_h x^{h+d}(Bx)^*$. Thus we obtain, for h large enough

$$T^{(h)} = U^{(h)}(1 - H) - G_d a_h x^{h+d}(Bx)^* - P_h$$

with $P_h \in K_+[x]$. $\square$

Note that the assertion on star height 2 may also be obtained by drawing the automaton associated to the Perrin companion matrix, see [14], Fig. 1.

4. Notes

A proof of Theorem 1.1 based on the Perron–Frobenius theorem has been given by [6].

The proof of Theorem 2.1 given here is based on [15,14] with some new ideas, in particular Lemmas 2.3 and 2.4. The latter is reminiscent of a theorem of Handelman, see [8,14]. Recently, algorithmic aspects of the construction have been considered in [2] and in [11,12].

There is another proof of Soittola's theorem given in [10]. The proof works when the dominating eigenvalue is simple. However, when the eigenvalue is multiple, the proof seems to be incorrect. Indeed, with the notation of the paper, we get, from the equations in the proof of Lemma 6 of [10], the equation $g_{N+2} = a_{N+2} + (d_1 - \alpha_1)a_{N+1} = a_{N+2} - \alpha_1 a_{N+1} + d_1 a_{N+1}$. By the exponential polynomial in Equation (3.5) in [10], $a_{N+2} - \alpha_1 a_{N+1}$ grows as $N^{K_1-2}\alpha_1^N$ for $N \to \infty$, whereas the last term $d_1 a_{N+1}$ grows as $N^{K_1-1}\alpha_1^N$. Since d_1 is not shown to be positive (and there seems to be no reason for this to hold), g_{N+2} is not positive for sufficiently large N, contrary to what is asserted in page 92, line 2. A concrete counter-example to the proof of Lemma 6 in [10] is the following. Let $f(z) = z/P(z)$ with $P(z) = (1-\alpha z)(1-\alpha z)(1-z)$ (here $\alpha = \alpha_1$). Then we take, as in Lemma 5, $P_1(z) = (1-\alpha z)(1-z)$ and therefore $R_1(z) = 1-z$. It is easily seen that $f(z)$ has positive coefficients for $\alpha > 0$, and the hypothesis of Lemma 6 is fulfilled if α is chosen large enough. Finally, $d_1 = -1$. Moreover, taking $\alpha = 2$, we find that $g_{N+2} = -(N+2)2^{N+1} - 2$, as indicated by one of the referees.

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